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Some Properties of the Peak-Set-Mapping

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1. INTRODUCTION

Let X be a normed linear space over the field of the real or complex numbers and let X' be its dual space. We study the set-valued mapping

$$\Sigma(x) := \{L \in X' : \| L \| \leqslant 1 \text{ and } L(x) = \| x \|\},$$

which associates to each $x \in X$ the *peak set* $\Sigma(x)$.

First, we apply the known fact that Σ is upper semicontinuous [12, 3] to some problems in the theory of Chebyshev-approximation of continuous vector-values functions.

Let V be a subset of X and $x \in X$. An element $v_0 \in V$ is called a *best approximation* for x by elements of the set V whenever $\| x - v_0 \| = \min\{\| x - v \|: v \in V\}$. For $x \in X$, we denote by $P_V(x)$ the set of all best approximations for x by elements of V. The set-valued mapping P_V is called the *metric projection* associated with V.

For T a locally compact Hausdorff-space, $C_0(T, X)$ denotes the space of continuous functions f: $T \to X$ which vanish at infinity, that is for each $\epsilon > 0$ the set

$$\{t \in T : \| f(t) \| \geqslant \epsilon\}$$

is compact. This linear space is provided with the Chebyshev-norm

$$\| f \| := \max\{\| f(t) \|_X : t \in T\}.$$

(In order to distinguish between the norms in $C_0(T, X)$ and X we shall sometimes denote the latter by $\| \cdot \|_X$). If X is the real axis R and T a compact space we shall simply write $C(T)$ instead of $C_0(T, R)$.

We prove by a direct method which does not use the representation of the extreme points of the unit-ball in the dual space of $C_0(T, X)$, that the generalized Kolmogorov-criterion is always sufficient to ensure that $v_0 \in V$ be a best approximation for $f \in C_0(T, X)$. Then we characterize the so-called Kolmogorov-sets, that is, those sets for which the Kolmogorov-criterion is also necessary. In the limiting cases this characterization reduces to one which is known [4, 5, 6, 10]. A direct approach to such a characterization is desirable because the results for $C_0(T, X)$ are rather difficult to derive from the general results for normed linear spaces, cf. for instance [10], where such a derivation is performed for $C_0(T, H)$, where T is a compact and H a pre-Hilbert space.

In the second part of the paper we study spaces X which have the property that to every point $x_0 \in X$ there exist certain neighborhoods U and points $x_U \in U$ such that $\Sigma(x) \subset \Sigma(x_U)$ for all $x \in U$. We show that in spaces with this maximum property each moon is a Kolmogorov-set. (The concept of a moon was introduced and studied by Amir and Deutsch [1].) We show that the (A)-spaces of Brosowski and Deutsch [7] have this maximum property. Furthermore, we demonstrate that the (QP)-spaces of Amir and Deutsch [1] are characterized by the property that each $x_0 \in X$ has a neighborhood U such that $\Sigma(x) \subset \Sigma(x_0)$ for every $x \in U$. Finally, we point out that spaces which have this maximum property of the peak-set-mapping are strongly nonlunar in the sense of Amir and Deutsch [1].

In the third part we show how the (P)-spaces of Brown [11] are characterized by an elementary property of Σ. Furthermore, we demonstrate that in (P)-spaces the continuity of the set-valued metric projection $P_V(x)$ associated with a subset V of X, which was proved by Brown for linear subspaces only, also holds for certain convex approximatively compact sets V. An example will show that this continuity is not valid for an arbitrary convex compact set V.

2. The Upper Semicontinuity of Σ

We begin by recalling some definitions (cf. [2, p. 114]).

Definition 2.1. Let T and Y be topological spaces.

(a) A set-valued mapping $A: T \to Y$ is called upper semicontinuous if for each $t_0 \in T$ and each open set $U \subset Y$ with $A(t_0) \subset U$ there is a neighborhood V of t_0 such that $A(t) \subset U$ for all $t \in V$.

(b) A set-valued mapping $A: T \to Y$ is called lower semicontinuous if for each $t_0 \in T$ and each open set $U \subset Y$ with $A(t_0) \cap U \neq \varnothing$ there is a neighborhood V of t_0 such that $A(t) \cap U \neq \varnothing$ for all $t \in V$.

The terms upper and lower semicontinuous will be abbreviated by usc and lsc, respectively. A set-valued mapping $A: T \to Y$ is called continuous if it is usc and lsc.

We assume throughout this paper that the closed unit ball

$$B(X') := \{L \in X' : \|L\| \leq 1\}$$

in the dual space X' is endowed with the weak topology $\sigma(X', X)$. Hence, $B(X')$ is compact.

LEMMA 2.2. *The bilinear function* $\Psi(x, L) := L(x)$ *is continuous on* $X \times B(X')$.

Proof. Let $x_0 \in X$, $L_0 \in B(X')$ and $\epsilon > 0$. Define

$$U := \{x \in X: \|x - x_0\| < \epsilon/2\}$$

and

$$V := \{L \in B(X') : |L(x_0) - L_0(x_0)| < \epsilon/2\}.$$

Then U and V are neighborhoods of x_0 and L_0, respectively. For $(x, L) \in U \times V$, we obtain

$$\begin{aligned} |\Psi(x, L) - \Psi(x_0, L_0)| &= |L(x) - L_0(x_0)| \\ &\leq |L(x) - L(x_0)| + |L(x_0) - L_0(x_0)| \\ &< \|x - x_0\| + \epsilon/2 < \epsilon. \end{aligned}$$

COROLLARY 2.3. *Let T be a topological space, and $f: T \to X$ a continuous mapping. Then the function $\Phi(t, L) := L(f(t))$ is continuous on $T \times B(X')$.*

LEMMA 2.4. *The set-valued mapping $\Sigma: X \to B(X')$, which is defined by*

$$\Sigma(x) := \{L \in B(X') : L(x) = \|x\|\},$$

is usc.

Proof. By virtue of (2.2) the function $F(x, L) := L(x) - \|x\|$ is continuous on $X \times B(X')$. The graph Γ of Σ may be represented by

$$\begin{aligned} \Gamma &:= \{(x, L) \in X \times B(X') : L \in \Sigma(x)\} \\ &= \{(x, L) \in X \times B(X') : F(x, L) = 0\}. \end{aligned}$$

Hence, Γ is closed, and Σ is usc (cf. [2, p. 118]).

From the continuity properties of a set-valued mapping follow continuity properties for some associated real functions as demonstrated in the next lemma, which is a consequence of a theorem of Berge [2, p. 122, Theorem 2]. For single-valued real functions we use the usual semicontinuity concepts which must not be confused with those of definition (2.1).

LEMMA 2.5. *Let T be a topological space, A: $T \to B(X')$ an* usc *set-valued mapping such that $A(t)$ is nonvoid and compact for every $t \in T$, and let f: $T \to X$ be a continuous function. Then the real functions*

$$\phi(t) := \min\{\text{Re } L(f(t)) : L \in A(t)\},$$
$$\psi(t) := \max\{\text{Re } L(f(t)) : L \in A(t)\}$$

are lsc *and* usc, *respectively.*

From (2.4) and (2.5) we may conclude the following corollary immediately.

COROLLARY 2.6. *Let T be a topological space and $f, g : T \to X$ continuous functions. Then the real function*

$$\phi(t) := \min\{\text{Re } L(f(t)) : L \in \Sigma(g(t))\}$$

is lsc.

3. THE GENERALIZED KOLMOGOROV CRITERION IN $C_0(T, X)$

By $\mathscr{E}(X')$ we denote the set of extreme points of $B(X')$ and, for $x \in X$, by $\mathscr{E}(x) := Ep(\Sigma(x))$ the set of the extreme points of $\Sigma(x)$. For $f \in C_0(T, X)$, we define

$$M(f) := \{t \in T : \|f(t)\|_X = \|f\|\}.$$

Best approximations in a normed linear space X may be characterized by use of Singer's generalization of the classical *Kolmogorov criterion* [14, p. 62]: Let V be a linear subspace of X, $x \in X \setminus \overline{V}$ and $v_0 \in V$. Then v_0 is in $P_V(x)$ if and only if

$$\min\{\text{Re } L(v - v_0) : L \in \mathscr{E}(x - v_0)\} \leqslant 0, \qquad \text{for all} \quad v \in V.$$

The application of this criterion in the space $C_0(T, X)$ requires knowledge of the extremal functionals on $C_0(T, X)$. Such knowledge is rather difficult to obtain. In the case T compact and X a Banach space Singer [14, p. 191], and in the general case Brosowski and Deutsch [7], have proved that a linear functional Λ on $C_0(T, X)$ is extremal if and only if there is a point $t_0 \in T$ and an extreme point L_0 of $B(X')$ such that

$$\Lambda(f) = L_0(f(t_0)) \qquad \text{for all} \quad f \in C_0(T, X).$$

We have mentioned these results in order to show that the following results, which we arrive at by a direct approach, fit in the general framework of theorems of characterization in normed linear spaces.

At first we state the sufficiency part of the generalized Kolmogorov criterion.

THEOREM 3.1. *Let V be a subset of $C_0(T, X)$, and f and v_0 elements of $C_0(T, X)\setminus V$ and V, respectively. Whenever for each $v \in V$ the inequality*

$$\min\{\operatorname{Re} L(v(t) - v_0(t)) : t \in M(f - v_0) \text{ and } L \in \mathscr{E}(f(t) - v_0(t))\} \leqslant 0 \quad (1)$$

holds, then v_0 is a best approximation for f by elements of V.

Remark. At first we show that the minimum in (1) is always attained. Since the set

$$\Gamma := \{(t, L) \in T \times B(X') : t \in M(f - v_0) \text{ and } L \in \Sigma(f(t) - v_0(t))\}$$

is a compact subset of $T \times B(X')$, and the function

$$\Phi(t, L) := \operatorname{Re} L(v(t) - v_0(t))$$

is continuous on $T \times B(X')$, there exists $(t_0, L_0) \in \Gamma$ such that

$$\Phi(t_0, L_0) = \min\{\Phi(t, L) : (t, L) \in \Gamma\}.$$

The set

$$\{L_1 \in \Sigma(f(t_0) - v_0(t_0)) : \Phi(t_0, L_1) = \min\{\Phi(t_0, L) : L \in \Sigma(f(t_0) - v_0(t_0))\}\}$$

is a nonvoid extremal subset of $\Sigma(f(t_0) - v_0(t_0))$. Therefore, it contains extreme points. These are also extreme points of $\Sigma(f(t_0) - v_0(t_0))$. Thus, the functional L_0 may be chosen in $\mathscr{E}(f(t_0) - v_0(t_0))$, and the minimum in (1) is attained.

Proof of theorem 3.1. Suppose that v_0 is not a best approximation for f. Then there exists an element $v \in V$ such that

$$\|f(t) - v(t)\|_X < \|f - v_0\| \qquad \text{for all} \quad t \in T. \qquad (2)$$

By (1), there is $t_0 \in M(f - v_0)$ and $L_0 \in \mathscr{E}(f(t_0) - v_0(t_0))$ with

$$\operatorname{Re} L_0(v(t_0) - v_0(t_0)) \leqslant 0.$$

Hence, it follows that

$$\begin{aligned}\|f(t_0) - v(t_0)\|_X &\geqslant \operatorname{Re} L_0(f(t_0) - v(t_0)) \\ &= \operatorname{Re} L_0(f(t_0) - v_0(t_0)) - \operatorname{Re} L_0(v(t_0) - v_0(t_0)) \\ &= \|f - v_0\| - \operatorname{Re} L_0(v(t_0) - v_0(t_0)) \geqslant \|f - v_0\|,\end{aligned}$$

which contradicts the supposed inequality (2).

It is well known that the Kolmogorov criterion is always necessary for convex but not for arbitrary sets V. Those sets V for which the Kolmogorov criterion is always necessary are called *Kolmogorov sets* [10]. In order to describe these sets geometrically Brosowski [4, 6] and Brosowski and Wegmann [10] introduced the concept of regular sets. So as to avoid further misuse of the term "regular" we introduce here instead a property (R).

DEFINITION 3.2. Let V be a subset of $C_0(T, X)$. We say $v_0 \in V$ is an (R)-point of V if the following requirement is fulfilled: Whenever for elements $f \in C_0(T, X)$, $v \in V$, and an usc set-valued mapping $A: T \to B(X')$, which takes values in the set of closed subsets of $B(X')$ such that $A(t) \supset \mathscr{E}(f(t))$ for $t \in M(f)$ the inequality

$$\operatorname{Re} L(v(t) - v_0(t)) > 0$$

holds for all $t \in M(f)$ and $L \in A(t) \cap \mathscr{E}(X')$ then there exists for every $\lambda > 0$ an element $v_\lambda \in V$ such that $\| v_\lambda - v_0 \| < \lambda$ and

$$\operatorname{Re} L(v_\lambda(t) - v_0(t)) > \operatorname{Re} L(f(t)) - \| f \|$$

for $t \in M(f)$ and $L \in A(t) \cap \mathscr{E}(X')$.

LEMMA 3.3. *Let V be a subset of $C_0(T, X)$, v_0 an (R)-point of V and $f \in C_0(T, X)$. Whenever for an element $v \in V$ the inequality*

$$\operatorname{Re} L(v(t) - v_0(t)) > 0 \quad \textit{holds for} \quad t \in M(f - v_0) \quad \textit{and} \quad L \in \mathscr{E}(f(t) - v_0(t)),$$

then there is for every $\lambda > 0$ an element $v_\lambda \in V$ such that $\| v_0 - v_\lambda \| < \lambda$ and $\| f - v_\lambda \| < \| f - v_0 \|$.

Proof. From the assumption it follows that even $\operatorname{Re} L(v(t) - v_0(t)) > 0$ for $t \in M(f - v_0)$ and $L \in \Sigma(f(t) - v_0(t))$. The set $M(f - v_0)$ is compact, and the function

$$\phi(t) := \min\{\operatorname{Re} L(v(t) - v_0(t)) : L \in \Sigma(f(t) - v_0(t))\} \tag{3}$$

is lcs. by virtue of (2.6). Therefore, we have $\phi(t) \geqslant a > 0$ for $t \in M(f - v_0)$.

The open set $U_1 := \{t \in T : \phi(t) > a/2\}$ contains $M(f - v_0)$. Since $v - v_0$ vanishes at infinity, and $\phi(t) \leqslant \| v(t) - v_0(t) \|_X$, the set U_1 is relatively compact. U_1 is a neighborhood of the compact set $M(f - v_0)$. Since a locally compact space is a regular space, the closed neighborhoods of a compact set form a basis of neighborhoods. Hence, there exists an open set U_2 such that

$$M(f - v_0) \subset U_2 \subset \overline{U}_2 \subset U_1 .$$

For $t \in T \setminus U_2$, we have

$$\|f(t) - v_0(t)\|_X < \|f - v_0\|. \tag{4}$$

From the inclusion $\overline{U}_2 \subset U_1$ it follows that

$$\operatorname{Re} L(v(t) - v_0(t)) > a/2 > 0 \tag{5}$$

holds for $t \in \overline{U}_2$ and $L \in \Sigma(f(t) - v_0(t))$. Hence, $v(t) - v_0(t) \neq 0$ and $f(t) - v_0(t) \neq 0$ for all $t \in \overline{U}_2$.

Since $\overline{U}_2$ is compact,

$$a_1 := \min\{\|f(t) - v_0(t)\|_X : t \in \overline{U}_2\}$$

is positive. The function

$$\rho_1(t) := \min(1, a_1/\|f(t) - v_0(t)\|_X)$$

is continuous. (For t with $\|f(t) - v_0(t)\|_X = 0$ put $\rho_1(t) = 1$.) Since ϕ as defined in (3) is lsc, the set

$$C := \{t \in T : \phi(t) \leqslant 0\}$$

is closed, and $C \cap U_1 = \varnothing$.

There exists a continuous function ρ_2 such that $0 \leqslant \rho_2(t) \leqslant 1$, $\rho_2(t) = 1$ for $t \in \overline{U}_2$ and $\rho_2(t) = 0$ outside U_1 (cf. [13, p. 247]). Then in particular $\rho_2(t)$ vanishes for $t \in C$.

Now define $\rho(t) := \rho_1(t) \cdot \rho_2(t)$ and $f_1(t) := v_0(t) + \rho(t)(f(t) - v_0(t))$ for $t \in T$. By construction, we obtain $f_1 \in C_0(T, X)$, $T \setminus C \supset M(f_1 - v_0) \supset \overline{U}_2$ and $\Sigma(f_1(t) - v_0(t)) = \Sigma(f(t) - v_0(t))$ for those t which have $f_1(t) - v_0(t) \neq 0$. Hence, it follows that $\operatorname{Re} L(v(t) - v_0(t)) > 0$ for $t \in M(f_1 - v_0)$ and $L \in \Sigma(f_1(t) - v_0(t))$.

By virtue of the lower semicontinuity of the function

$$\phi_1(t) := \min\{\operatorname{Re} L(v(t) - v_0(t)) : L \in \Sigma(f_1(t) - v_0(t))\}$$

we have $\phi_1(t) \geqslant b > 0$, for $t \in M(f_1 - v_0)$ and $L \in \Sigma(f_1(t) - v_0(t))$. For every $t \in M(f_1 - v_0)$, the set

$$A(t) := \{L \in B(X') : \operatorname{Re} L(v(t) - v_0(t)) \geqslant b/2\}$$

is closed, and contains $\mathscr{E}(f_1(t) - v_0(t))$. The set-valued mapping $A: T \to B(X')$ is usc since the function $\Phi(t, L) := \operatorname{Re} L(v(t) - v_0(t))$ is continuous on $T \times B(X')$, by (2.3), and the graph of A

$$\Gamma(A) := \{(t, L) \in T \times B(X') : \Phi(t, L) \geqslant b/2\}$$

is closed. Now, since v_0 is an (R)-point of V, there exists for every $\lambda > 0$ an element $v_\lambda \in V$ such that $\|v_\lambda - v_0\| < \lambda$ and

$$\text{Re } L(v_\lambda(t) - v_0(t)) > \text{Re } L(f_1(t) - v_0(t)) - \|f_1 - v_0\|$$
$$\text{for} \quad t \in M(f_1 - v_0) \quad \text{and} \quad L \in A(t) \cap \mathscr{E}(X'). \quad (6)$$

Since $\|f(t) - v_0(t)\|_X$ vanishes at infinity and $T\setminus U_2$ is closed and disjoint from $M(f - v_0)$, it follows that

$$e_1 := \max\{\|f(t) - v_0(t)\|_X : t \in T\setminus U_2\} < \|f - v_0\|$$

The set-valued mapping

$$A_1(t) := \{L \in B(X') : \text{Re } L(v(t) - v_0(t)) \geq h/2\}$$

is usc, by the same reasons as A. Since

$$A_1(t) \cap \Sigma(f_1(t) - v_0(t)) = \varnothing \quad \text{for} \quad t \in M(f_1 - v_0)$$

the function

$$\phi_2(t) := \max\{\text{Re } L(f_1(t) - v_0(t)) : L \in A_1(t)\}$$

is such that

$$\phi_2(t) < \|f_1(t) - v_0(t)\|_X = \|f_1 - v_0\|$$

for $t \in M(f_1 - v_0)$. By (2.5) ϕ_2 is usc. So we have

$$e_2 := \max\{\phi_2(t) : t \in M(f_1 - v_0)\} < \|f_1 - v_0\|.$$

Choose λ so that

$$0 < \lambda < \min(\|f - v_0\| - e_1, \|f_1 - v_0\| - e_2).$$

Then we obtain for $t \in T\setminus U_2$

$$\|f(t) - v_\lambda(t)\|_X \leq \|f(t) - v_0(t)\|_X + \|v_0(t) - v_\lambda(t)\|_X$$
$$\leq e_1 + \lambda < \|f - v_0\|.$$

Now let t be in U_2. Then t is in $M(f_1 - v_0)$, since $U_2 \subset M(f_1 - v_0)$. For $L \in A(t) \cap \mathscr{E}(X')$ we obtain by (6)

$$\text{Re } L(f_1(t) - v_\lambda(t)) = \text{Re } L(f_1(t) - v_0(t)) + \text{Re } L(v_0(t) - v_\lambda(t))$$
$$< \|f_1 - v_0\|.$$

On the other hand, for $L \in A_1(t)$ the inequality

$$\operatorname{Re} L(f_1(t) - v_\lambda(t)) = \operatorname{Re} L(f_1(t) - v_0(t)) + \operatorname{Re} L(v_0(t) - v_\lambda(t))$$
$$\leqslant \phi_2(t) + \lambda < \| f_1 - v_0 \|$$

holds.

Since $(A(t) \cap \mathscr{E}(X')) \cup A_1(t) \supset \mathscr{E}(X')$, it follows that

$$\| f_1(t) - v_\lambda(t) \|_X < \| f_1 - v_0 \|.$$

By construction of f_1, we have

$$f_1(t) - v_0(t) = \rho(t)(f(t) - v_0(t)),$$

with a function $\rho(t)$ such that $0 \leqslant \rho(t) \leqslant 1$. Therefore, for every $t \in U_2$, we obtain

$$\| f(t) - v_0(t) \|_X = \| f(t) - f_1(t) \|_X + \| f_1(t) - v_0(t) \|_X$$
$$> \| f(t) - f_1(t) \|_X + \| f_1(t) - v_\lambda(t) \|_X \geqslant \| f(t) - v_\lambda(t) \|_X$$

which yields finally

$$\| f(t) - v_\lambda(t) \|_X < \| f(t) - v_0(t) \|_X \qquad \text{for all} \quad t \in T.$$

We are now ready to give the promised characterization of Kolmogorov sets in $C_0(T, X)$.

THEOREM 3.4. *A set $V \subset C_0(T, X)$ is a Kolmogorov set if and only if each point v_0 of V is an* (R)*-point of V.*

Proof. Let V be a Kolmogorov set. Suppose that there exists some $v_0 \in V$ which is not an (R)-point. That is, there exist $f \in C_0(T, X)$, $v \in V$ and an usc set-valued mapping A: $T \to B(X')$, which takes values in the set of closed subsets of $B(X')$ with $A(t) \supset \mathscr{E}(f(t) - v_0(t))$, for $t \in M(f - v_0)$, such that

$$\operatorname{Re} L(v(t) - v_0(t)) > 0$$
$$\text{for} \quad t \in M(f - v_0) \quad \text{and} \quad L \in A(t) \cap \mathscr{E}(X'), \tag{6a}$$

and there exists $\lambda > 0$ such that, for every $v_\lambda \in V$ with $\| v_\lambda - v_0 \| < \lambda$, there exist $t_0 \in M(f - v_0)$ and $L_0 \in A(t) \cap \mathscr{E}(X')$ so that

$$\operatorname{Re} L_0(v_\lambda(t_0) - v_0(t_0)) \leqslant \operatorname{Re} L_0(f(t_0) - v_0(t_0)) - \| f(t_0) - v_0(t_0) \|_X .$$

Hence, it follows that

$$\| f - v_0 \| = \| f(t_0) - v_0(t_0) \|_X \leqslant \operatorname{Re} L_0(f(t_0) - v_\lambda(t_0))$$
$$\leqslant \| f(t_0) - v_\lambda(t_0) \|_X \leqslant \| f - v_\lambda \|,$$

and, therefore, v_0 is a best approximation for f by elements of $V_\lambda := \{v \in V : \| v - v_0 \| < \lambda\}$.

Since V is a Kolmogorov set, each local minimum of the function $\Psi(v) := \| f - v \|$ on V is a global one (cf. [10, p. 382]) and the Kolmogorov criterion is necessary, implying that

$$\min\{\operatorname{Re} L(v(t) - v_0(t)) : t \in M(f - v_0),$$
$$L \in \mathscr{E}(f(t) - v_0(t))\} \leqslant 0 \qquad \text{for every } v \in V.$$

This contradicts (6a). Hence, v_0 must be an (R)-point.

Now let each point of V be an (R)-point. Suppose V is not a Kolmogorov set. Then there exist $f \in C_0(T, X)$ and $v_0 \in V$ such that v_0 is a best approximation for f, but there is a $v \in V$ such that

$$\operatorname{Re} L(v(t) - v_0(t)) > 0$$
$$\text{for} \quad t \in M(f - v_0) \quad \text{and} \quad L \in \mathscr{E}(f(t) - v_0(t)).$$

Since v_0 is an (R)-point, by Lemma 3.3, there is an element $v_\lambda \in V$ with $\| f - v_\lambda \| < \| f - v_0 \|$, that is, v_0 is not a best approximation for f by elements of V. This contradicts our assumption. Thus, V has to be a Kolmogorov set.

So as to give an application of Definition 3.2 and to point out that the most familiar Kolmogorov sets, the convex sets, are easily detected by means of Theorem 3.4; we now prove the following lemma.

LEMMA 3.5. *Let V be a subset of $C_0(T, X)$ which is star-shaped about some point $v_0 \in V$. Then v_0 is an* (R)*-point of V.*

Proof. Let $f \in C_0(T, X)$, $v \in V$, and let $A: T \to B(X')$ be a set-valued mapping with the properties required in Definition 3.2. For $\lambda > 0$, choose μ such that

$$0 < \mu < \min(\| v - v_0 \|, \lambda)$$

and define

$$v_\lambda := \left(1 - \frac{\mu}{\| v - v_0 \|}\right) v_0 + \frac{\mu}{\| v - v_0 \|} v.$$

Since V is star-shaped about v_0, the element v_λ is in V. By construction, we obtain

$$\| v_\lambda - v_0 \| = \mu < \lambda$$

and

$$\operatorname{Re} L(v_\lambda(t) - v_0(t)) = \frac{\mu}{\| v - v_0 \|} \operatorname{Re} L(v(t) - v_0(t))$$
$$> 0 = \operatorname{Re} L(f(t)) - \| f \|$$
$$\text{for} \quad t \in M(f) \quad \text{and} \quad L \in A(t) \cap \mathscr{E}(X').$$

Hence, v_0 is an (R)-point of V.

Now we consider special cases of Definition 3.2. If T consists of just one point then $C_0(T, X)$ is equal to X and Definition 3.2 of an (R)-point of a subset V of $C_0(T, X)$ reduces exactly to that of a regular point v_0 of V interpreted as a subset of X as introduced in [10].

For H a pre-Hilbert space with inner product $(\cdot, \cdot)$ and T a compact space, the concept of a regular point v_0 of a subset V of $C_0(T, H)$ was stated by Brosowski [4, 6] as follows: A point $v_0 \in V$ is called *regular* if and only if, for every $v \in V$, for every closed subset $C \subset T$ and for every continuous function f: $C \to H$ with

$$\operatorname{Re}(f(t), v(t) - v_0(t)) > 0 \qquad \text{for} \quad t \in C,$$

there exists, for each $\lambda > 0$, an element $v_\lambda \in V$ such that $\| v_\lambda - v_0 \| < \lambda$ and

$$2 \operatorname{Re}(f(t), v_\lambda(t) - v_0(t)) > \| v_\lambda(t) - v_0(t) \|_H^2 \qquad \text{for} \quad t \in C.$$

First, we note that for $x \in H$ the peak set $\Sigma(x)$ consists of the only functional L defined by

$$L(y) := (y, x/\| x \|) \qquad \text{for} \quad y \in H.$$

Now let v_0 be a regular point of V, and let $f \in C_0(T, H)$, $v \in V$, and let A: $T \to B(H')$ be an usc set-valued mapping such that $A(t) \supset \mathscr{E}(f(t))$ and $\operatorname{Re} L(v(t) - v_0(t)) > 0$ for $t \in M(f)$ and $L \in A(t) \cap \mathscr{E}(H')$. Then in particular $\operatorname{Re}(f(t), v(t) - v_0(t)) > 0$ holds for $t \in M(f)$, and since v_0 is a regular point of V there exists for every $\lambda > 0$ a $v_\lambda \in V$ so that $\| v_0 - v_\lambda \| < \lambda$ and

$$2 \operatorname{Re}(f(t), v_\lambda(t) - v_0(t)) > \| v_\lambda(t) - v_0(t) \|_H^2 \qquad \text{for} \quad t \in M(f).$$

This is equivalent to

$$\| f(t) + v_0(t) - v_\lambda(t) \|_H < \| f(t) \|_H \qquad \text{for each } t \in M(f).$$

Therefore,

$$\operatorname{Re} L(v_\lambda(t) - v_0(t)) > \operatorname{Re} L(f(t)) - \| f \|$$

holds for all $t \in M(f)$ and $L \in B(H')$, in particular for the functionals $L \in A(t) \cap \mathscr{E}(H')$ as required in Definition 3.2.

The proof of the converse statement, that each (R)-point in $C_0(T, H)$ is a regular point is more difficult and makes use of parts of the proof of Lemma 3.3. Although both concepts are equivalent, the notion of an (R)-point is formally weaker than that of a regular point.

4. Maximum Properties of Σ

In the Euclidean R^2 the only sets U on which $\Sigma(x)$ attains a greatest value in the $\subset$-ordering are the sets with $0 \in U$ and the sets which consist of positive mutiples of one single vector. But there are spaces where each point admits a basis of neighborhoods U which contain points x_U with $\Sigma(x) \subset \Sigma(x_U)$ for all $x \in U$. The space R^2 with the Chebyshev-norm is a simple example. We demonstrate that these spaces have some interest in investigations in "moons" recently done by Amir, Brosowski, and Deutsch [1, 7].

We recall some definitions and introduce some notations. For x and v_0 in X we define the open cone

$$K(v_0, x) := \{v \in X : \operatorname{Re} L(v - v_0) > 0 \text{ for } L \in \Sigma(x - v_0)\}.$$

Furthermore, we define the unit sphere in X

$$S(X) := \{x \in X : \|x\| = 1\}$$

and the open ball

$$B(x_0, \epsilon) := \{x \in X : \|x - x_0\| < \epsilon\}$$

with center x_0 and radius ϵ.

Let V be a subset of X. An element $v_0 \in V$ is called a *lunar point* if $v_0 \in \overline{V \cap K(v_0, x)}$ whenever $x \in X$ and $V \cap K(v_0, x) \neq \varnothing$. The set V is called a *moon* if each of its points is lunar.

The definition of a Kolmogorov set reads in this context as follows: V is a Kolmogorov set if and only if $V \cap K(v_0, x) = \varnothing$ whenever $v_0 \in V$ is a best approximation for x.

Each Kolmogorov set is a moon. The converse is not true in general. It has been noted [1, 7] that in certain familiar spaces, such as $C(T)$ and l_1, each moon is a Kolmogorov set. Now we give a description of these spaces in terms of a property (MP) ("*maximum peak set*") of Σ.

DEFINITION 4.1. A space X is said to have property (MP) if for each $x_0 \in X$ there exists a system $\mathfrak{A}$ of neighborhoods U of x_0 such that

(a) For every $U \in \mathfrak{A}$ there is $x_U \in U$ with

$$\Sigma(x) \subset \Sigma(x_U) \qquad \text{for all } x \in U.$$

(b) For every subset $W \subset X'$ which is a neighborhood of $\Sigma(x_0)$ in the topology $\sigma(X', X)$ there exists U in $\mathfrak{A}$ such that $\Sigma(x_U) \subset W$.

For the remainder of this section we assume that the spaces X are over the field of the real numbers, because no spaces over the complex field have property (MP). To show this, let X have property (MP), and let $x_0 \neq 0$ be an element of X. Then there exists a neighborhood U of x_0 such that $x_U \neq 0$ and $\Sigma(x) \subset \Sigma(x_U)$ for all x in U. If X is over the complex field, there exists some scalar α with $\operatorname{Im}(\alpha) \neq 0$ such that $\alpha x_0 \in U$. Since for $L \in \Sigma(\alpha x_0)$ the equality $L(\alpha x_0) = |\alpha| \cdot \|x_0\|$ holds, the functional $L_1 := \alpha/|\alpha| \cdot L$ is in $\Sigma(x_0)$. Hence, we obtain the equality

$$\|x_U\| = L_1(x_U) = \frac{\alpha}{|\alpha|} L(x_U) = \frac{\alpha}{|\alpha|} \|x_U\|,$$

which is impossible for $x_U \neq 0$.

THEOREM 4.2. *If X has the property* (MP) *then every moon is a Kolmogorov set.*

Proof. Let V be a moon and $v_0 \in V$ a best approximation for $f \in X$ by elements of V. We have to show

$$\min\{L(v - v_0) : L \in \mathscr{E}(f - v_0)\} \leq 0$$

for every $v \in V$. Suppose that there is a v in V such that $L(v - v_0) > 0$ for all $L \in \mathscr{E}(f - v_0)$. Then $L(v - v_0) > 0$ for all $L \in \Sigma(f - v_0)$ and $L(v - v_0) \geq a > 0$ since $\Sigma(f - v_0)$ is compact. The set

$$W := \{L \in X' : L(v - v_0) > a/2\}$$

is an open neighborhood of $\Sigma(f - v_0)$. Since X has property (MP) there is a neighborhood U of $f - v_0$ and $g_U \in U$ such that $\Sigma(g) \subset \Sigma(g_U) \subset W$ for all $g \in U$. For $f_1 := v_0 + g_U$, it follows that

$$L(v - v_0) > 0 \qquad \text{for all } L \in \Sigma(f_1 - v_0),$$

that is $v \in K(v_0, f_1)$. Since V is a moon there exists for every $\lambda > 0$ an element $v_\lambda \in V$ such that $\|v_\lambda - v_0\| < \lambda$ and

$$L(v_\lambda - v_0) > 0 \qquad \text{for all } L \in \Sigma(f_1 - v_0). \tag{7}$$

We choose $\lambda > 0$ so small that $B(f - v_0, \lambda) \subset U$, and show that $\|f - v_\lambda\| < \|f - v_0\|$.

Suppose that $\|f - v_\lambda\| \geq \|f - v_0\|$. Then the outer part

$$Z_0 := \{v \in Z : \|f - v\| \geq \|f - v_0\|\}$$

of the segment

$$Z := \{v_\lambda + \theta(v_0 - v_\lambda) : 0 \leq \theta < 1\}$$

is convex and, since $v_\lambda \in Z_0$ by assumption, nonvoid. There exists a functional, which separates Z_0 from the ball $B(f, \|f - v_0\|)$, that is, there exists $L_0 \in X'$ such that $\| L_0 \| = 1$ and

$$\sup\{L_0(v) : v \in Z_0\} \leq \inf\{L_0(v) : v \in B(f, \|f - v_0\|)\},$$

whence it follows that

$$L_0(v_\lambda - v_0) \leq 0. \tag{8}$$

Define $v_\theta := v_\lambda + \theta(v_0 - v_\lambda)$ and

$$\theta_0 := \min\{\theta \in [0, 1] : \|f - v_\theta\| = \|f - v_0\|\}.$$

Then we have

$$\begin{aligned} L_0(v_{\theta_0} - f) &= \inf\{L_0(v - f) : v \in B(f, \|f - v_0\|)\} = - \|f - v_0\| \\ &= - \|f - v_{\theta_0}\|, \end{aligned}$$

and, therefore, $L_0 \in \Sigma(f - v_{\theta_0})$. Since $v_{\theta_0} \in B(v_0, \lambda)$, the element $f - v_{\theta_0}$ is in U and the functional L_0 is in $\Sigma(f_1 - v_0)$, and (8) contradicts (7). Therefore, we have $\|f - v_\lambda\| < \|f - v_0\|$, which contradicts the fact that v_0 is a best approximation for f by elements of V. Thus, the theorem is proved.

Now we give two examples of spaces with property (MP). Let first T be a compact Hausdorff space. We show that $X := C(T)$ has property (MP). Let $f \neq 0$ be in $C(T)$ and define $M(f) := \{t \in T : |f(t)| = \|f\|\}$. The peak set $\Sigma(f)$ consists of all Radon measures μ on T with $\|\mu\| = 1$, $\text{supp}(\mu) \subset M(f)$, and $\int f \, d\mu = \|f\|$. For ϵ with $0 < \epsilon < \|f\|$ we define

$$\begin{aligned} f_1(t) &:= \min(f(t) + \epsilon, \|f\| - \epsilon) \\ f_2(t) &:= \max(f(t) - \epsilon, - \|f\| + \epsilon). \end{aligned}$$

There is a continuous function ρ such that $0 \leq \rho(t) \leq 1$,

$$\rho(t) = 1 \qquad \text{for} \quad t \in \{t \in T : f_1(t) = \|f\| - \epsilon\}$$

and

$$\rho(t) = 0 \qquad \text{for} \quad t \in \{t \in T : f_2(t) = - \|f\| + \epsilon\}.$$

The function

$$g_U(t) := \rho(t) f_1(t) + (1 - \rho(t)) f_2(t)$$

has the property that $M(g_U) \supset M(g)$ for each $g \in U_\epsilon := \{g \in X : \|g - f\| \leq \epsilon\}$. Hence, $\Sigma(g) \subset \Sigma(g_U)$ for all $g \in U_\epsilon$. Because the sets U_ϵ form a basis of neighborhoods of the point f and the mapping Σ is usc, the requirements (a) and (b) of Definition 4.1. are proved.

Let now X be the space l_1 of all sequences $x = (x_i)$ which are absolutely summable, normed by $\|x\| := \sum_{i=1}^{\infty} |x_i|$. The dual space X' is identical with l_∞, the space of bounded sequences. For $f \in X$, $\Sigma(f)$ is the set of all h in l_∞ such that $h_i = \text{sign}(f_i)$ for all i with $f_i \neq 0$. For $\epsilon > 0$ define

$$U_\epsilon := \{g \in l_1 : \|f - g\| \leq \epsilon\} \cup \{g_U\},$$

where g_U is given by

$$g_{U,i} := \begin{cases} f_i & \text{if } |f_i| > \epsilon, \\ 0 & \text{if } |f_i| \leq \epsilon. \end{cases}$$

Whenever $g_{U,i} \neq 0$ for an index $i \in N$, then $\text{sign}(g_i) = \text{sign}(g_{U,i})$ for all $g \in U_\epsilon$, whence it follows that $\Sigma(g) \subset \Sigma(g_U)$ for all $g \in U_\epsilon$. Since

$$\|f - g_U\| = \sum_{i=1}^{\infty} |f_i - g_{U,i}| = \sum_{|f_i| \leq \epsilon} |f_i|$$

converges to zero for $\epsilon \to 0$, the sets U_ϵ thus defined form a basis of neighborhoods of f, and l_1 has property (MP).

In the following the set $\mathscr{E}(X')$ of the extreme points of the unit ball in X' is provided with the topology which is induced by $\sigma(X', X)$. The interior of a subset A of $\mathscr{E}(X')$ in this topology is denoted by $\mathring{A}$.

Brosowski and Deutsch [7] introduced a *property* (A) as follows: A normed linear space X is said to have property (A) if for each $f \neq 0$ in X there is a family $(g_\tau)_{\tau \in T}$ of elements g_τ in X such that

(1) For each open subset W of $\mathscr{E}(X')$ which contains $\mathscr{E}(f)$ there is a $\tau \in T$ with $\mathscr{E}(g_\tau) \subset W$.

(2) For each $\tau \in T$,

$$\sup\{L(f) : L \in \mathscr{E}(X') \backslash \mathscr{E}(g_\tau)\} < \|f\|.$$

THEOREM 4.3. *Each space with property* (A) *has property* (MP).

Proof. Let f be an element in X and $W \subset X'$ a $\sigma(X', X)$-neighborhood of $\Sigma(f)$. Since each compact convex set in a locally convex space admits a basis of convex and open neighborhoods, we may assume that W is convex and open. Because X has property (A) there exists g_τ in X such that

$$\mathscr{E}(f) \subset \mathring{\mathscr{E}}(g_\tau) \subset \mathscr{E}(g_\tau) \subset W.$$

In spaces with property (A) the set-valued mapping $x \to \mathscr{E}(x)$ is usc (cf. [7]). Hence, the set

$$U := \{h \in X : \mathscr{E}(h) \subset \mathring{\mathscr{E}}(g_\tau)\}$$

is a neighborhood of f. We have $\mathscr{E}(g) \subset \mathscr{E}(g_\tau) \subset W$ for each $g \in U$ and, since W is a $\sigma(X', X)$-open convex subset of X', even $\Sigma(g) \subset \Sigma(g_\tau) \subset W$ for all $g \in U$.

Amir and Deutsch [1] called an element v_0 of the sphere $S(X)$ a *quasipolyhedral point* (or (QP)-point) if $v_0 \notin \overline{K(v_0, 0) \cap S(X)}$. The space X is called a *(QP)-space* if each v_0 in $S(X)$ is a (QP)-point. The following theorem shows that the (QP)-spaces have property (MP), and gives a characterization of (QP)-spaces in terms of a maximum property of Σ.

THEOREM 4.4. *A space X is a* (QP)*-space if and only if for each $x_0 \in X$ there exists an $\epsilon > 0$ such that $\Sigma(x) \subset \Sigma(x_0)$ for $x \in B(x_0, \epsilon)$.*

Proof. Let X be a (QP)-space and x_0 an element in X. For $x_0 = 0$ we have $\Sigma(x_0) = B(X')$ and the statement is trivial. Let now be $x_0 \neq 0$. Then $y_0 := x_0/\| x_0 \|$ is a (QP)-point of $S(X)$, that is, there exists $\delta > 0$ such that

$$B(y_0, \delta) \cap K(y_0, 0) \subset B(0, 1).$$

From $K(y_0, 0) \supset B(0, 1)$ it follows that generally

$$B(y_0, \delta) \cap K(y_0, 0) \supset B(y_0, \delta) \cap B(0, 1).$$

Thus, we obtain

$$B(y_0, \delta) \cap K(y_0, 0) = B(y_0, \delta) \cap B(0, 1). \tag{9}$$

Now let y be in $S(X) \cap B(y_0, \delta)$ and L_0 in $\Sigma(y)$. In view of (9) there exists $\eta > 0$ such that for either sign $y \pm \eta(y_0 - y)$ is in $S(X)$. Thus, we obtain

$$\begin{aligned} 1 = L_0(y) &\geqslant L_0(y) \pm \eta L_0(y_0 - y) \\ &= 1 \pm \eta L_0(y_0 - y), \end{aligned}$$

whence it follows that $L_0(y_0) = L_0(y) = 1$, that is $L_0 \in \Sigma(y_0)$. If we choose ϵ in such a way that $0 < \epsilon < \| x_0 \|$ and $\| x/\| x \| - y_0 \| < \delta$ for $\| x - x_0 \| < \epsilon$ we get the statement of the theorem.

To prove the inverse conclusion let x_0 be a point of $S(X)$. By hypothesis there is $\epsilon > 0$ such that $\Sigma(x) \subset \Sigma(x_0)$ for $x \in B(x_0, \epsilon)$. Now let

$$x \in K(x_0, 0) \cap B(x_0, \epsilon) \qquad \text{and} \qquad L \in \Sigma(x).$$

Since $\Sigma(x) \subset \Sigma(x_0)$ we obtain

$$\begin{aligned} 1 = \| x_0 \| = L(x_0) &= L(x) + L(x_0 - x) \\ &= \| x \| + L(x_0 - x) > \| x \|, \end{aligned}$$

that is $x \in B(0, 1)$. We have proved

$$K(x_0, 0) \cap B(x_0, \epsilon) \subset B(0, 1),$$

whence it follows that x_0 is a (QP)-point.

In [1] the notion of a strongly nonlunar space was introduced. A point v_0 of the unit sphere $S(X)$ is called *strongly nonlunar* if for each u in $K(v_0, 0)$ there is an x in $B(0, 1)$ such that $u \in K(v_0, x)$, and there exists $\epsilon > 0$ so that $B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1)$. The space X is called strongly nonlunar if each point $v_0 \in S(X)$ is strongly nonlunar. Since in strongly nonlunar spaces each moon is a Kolmogorov set [1, Theorem 2.18], Theorem 4.2 may also be proved by means of the following theorem.

THEOREM 4.5. *Whenever X has property* (MP) *then X is strongly nonlunar.*

Proof. Let v_0 be in $S(X)$ and u in $K(v_0, 0)$. This means $L(u - v_0) < 0$ for all L in $\Sigma(v_0)$. Since $\Sigma(v_0)$ is compact, it follows that $L(u - v_0) \leqslant a < 0$ for all $L \in \Sigma(v_0)$. The set

$$W := \{L \in X' : L(u - v_0) < a/2\}$$

is a neighborhood of $\Sigma(v_0)$. Property (MP) ensures the existence of a neighborhood U of v_0 and an element x_U in U such that $\Sigma(v) \subset \Sigma(x_U) \subset W$ for all $v \in U$. We put $x_1 := v_0 - x_U$ and obtain by construction $L(u - v_0) < 0$ for each $L \in \Sigma(x_U) = \Sigma(v_0 - x_1)$, whence $u \in K(v_0, x_1)$.

There exists $\epsilon > 0$ such that $B(v_0, \epsilon) \subset U$. Let v_1 be an element of $B(v_0, \epsilon) \cap K(v_0, x_1)$. This means $\| v_1 - v_0 \| < \epsilon$ and

$$L(v_1 - v_0) < 0 \qquad \text{for each } L \text{ in } \Sigma(v_0 - x_1). \tag{10}$$

We show that $\| v_1 \| < 1$. Suppose on the contrary $\| v_1 \| \geqslant 1$, put $v_\theta := v_0 + \theta(v_1 - v_0)$, and define

$$\theta_0 := \max\{\theta \in [0, 1] : \| v_\theta \| \leqslant 1\}.$$

Since $\|v_1\| \geqslant 1$, the closed segment $[v_{\theta_0}, v_1]$ is outside $B(0, 1)$. By the separation theorem, there exists a functional $L_0 \in X'$, $\|L_0\| = 1$, such that

$$\sup\{L_0(v) : v \in B(0, 1)\} \leqslant \inf\{L_0(v) : v \in [v_{\theta_0}, v_1]\}.$$

The supremum on the left side is equal to $\|L_0\| = 1$, and since v_{θ_0} is a common boundary point of $B(0, 1)$ and $[v_{\theta_0}, v_1]$, we obtain $L_0(v_{\theta_0}) = 1 = \|v_{\theta_0}\|$ that is $L_0 \in \Sigma(v_{\theta_0})$. By construction we have $v_{\theta_0} \in U$, whence it follows that $L_0 \in \Sigma(x_U) = \Sigma(v_0 - x_1)$. From $L_0(v_0) \leqslant 1$ and $L_0(v_1) \geqslant 1$ we obtain $L_0(v_1 - v_0) \geqslant 0$, which contradicts (10). Thus, $\|v_1\| < 1$ is proved.

So far the proof has only used properties of the set $\Sigma(v_0 - x_1)$. This set does not change if $v_0 - x_1$ is multiplied by a positive factor. Passing to

$$x := v_0 - \frac{\epsilon}{2 \cdot \|v_0 - x_1\|}(v_0 - x_1),$$

we obtain $K(v_0, x) = K(v_0, x_1)$, $x \in K(v_0, x_1)$, $\|x - v_0\| < \epsilon$, and an argument similar to that used in the preceding part of the proof yields $x \in B(0, 1)$. This completes the proof.

5. (P)-Spaces and the Continuity of the Set-Valued Metric Projection

Brown [11] called a normed linear space X a (P)-*space* if for each pair of elements $x, z \in X$ with $\|x + z\| \leqslant \|x\|$ there exist positive numbers c and b such that $\|y + cz\| \leqslant \|y\|$ for all y with $\|x - y\| < b$. The following theorem characterizes (P)-spaces in terms of a property of Σ.

Theorem 5.1. *X is a* (P)*-space if and only if for every $x_0, x_1 \in X$ the following requirement holds: If for all x in the open segment (x_0, x_1) the peak set $\Sigma(x)$ is a subset of the hyperplane $H(x_1 - x_0) := \{L \in X' : L(x_1 - x_0) = 0\}$ (which is orthogonal to $x_1 - x_0$) then there exists a neighborhood U of (x_0, x_1) such that $\Sigma(x) \subset H(x_1 - x_0)$ for all $x \in U$.*

Proof. Let first X be a (P)-space and $x_0, x_1 \in X$ such that $\Sigma(x) \subset H(x_1 - x_0)$ for all $x \in (x_0, x_1)$, and let x_2 be an element of (x_0, x_1). We define $z := \alpha(x_1 - x_0)$ with

$$0 < \alpha < \min(\|x_2 - x_0\|, \|x_2 - x_1\|)/\|x_1 - x_0\|.$$

Since $L(z) = 0$ for $L \in \Sigma(x_2 \pm z)$, we obtain for either seign

$$\begin{aligned}\|x_2\| &\geqslant \max\{\operatorname{Re} L(x_2) : L \in \Sigma(x_2 \pm z)\}\\ &= \max\{\operatorname{Re} L(x_2 \pm z) : L \in \Sigma(x_2 \pm z)\}\\ &= \|x_2 \pm z\|.\end{aligned}$$

Because X is a (P)-space there exist $c > 0$ and a neighborhood $U(x_2)$ of x_2 such that

$$\| y \pm cz \| \leqslant \| y \| \qquad \text{for} \quad y \in U(x_2). \tag{11}$$

For every fixed y, z the function $\psi(\gamma) := \| y + \gamma z \|$ is a convex function in γ. Hence, it follows by virtue of (11) that $\| y + \gamma z \| = \| y \|$ for all real γ with $| \gamma | \leqslant c$. That means that for each point $y \in U(x_2)$ there is a segment $S := (y - cz, y + cz)$, parallel to (x_0, x_1), such that $\| x \| = \| y \|$ for all $x \in S$. Hence, we obtain $\Sigma(y) \subset H(x_1 - x_0)$, and the set

$$U := \bigcup_{x_2 \in (x_0, x_1)} U(x_2)$$

is a neighborhood of (x_0, x_1) with the property that $\Sigma(x) \subset H(x_1 - x_0)$ for all $x \in U$.

Now we prove the converse implication. Let x_0, z be elements of X such that $\| x_0 + z \| \leqslant \| x_0 \|$. Without loss of generality, we may assume $z \neq 0$. We consider two cases.

Let first $\| x_0 + \frac{1}{2}z \| < \| x_0 \|$. Since the function

$$F(x) := \| x \| - \| x + \tfrac{1}{2}z \|$$

is continuous and $F(x_0) > 0$ by assumption, there is a neighborhood U of x_0 such that $F(x) > 0$ for every $x \in U$. Therefore, there exists $b > 0$ such that with $c = 1/2$ the inequality $\| y + cz \| < \| y \|$ holds for all y with $\| y - x_0 \| < b$.

Now we consider the second case $\| x_0 + \frac{1}{2}z \| = \| x_0 \|$. Put $x_1 := x_0 + z$ and $\psi(\gamma) := \| x_0 + \gamma z \|$. Then ψ is convex and takes the value $\| x_0 \|$ at the three positions $\gamma = 0, \frac{1}{2}, 1$; hence, ψ must be constant in the domain $0 \leqslant \gamma \leqslant 1$. Thus, each x in the segment (x_0, x_1) has the norm $\| x \| = \| x_0 \|$. Therefore, we have $\Sigma(x) \subset H(x_1 - x_0)$ for every x in (x_0, x_1).

By hypothesis there is a neighborhood U for (x_0, x_1) such that $\Sigma(x) \subset H(x_1 - x_0)$ for all $x \in U$. Now let x_2 be an element in (x_0, x_1). We choose positive numbers b and c in such a way that

$$\{x \in X : \| x - x_2 \| < 2b\} \subset U \qquad \text{and} \qquad c < b/\| z \|.$$

For each y with $\| y - x_2 \| < b$ the inequality

$$\| y + cz - x_2 \| \leqslant \| y - x_2 \| + c \| z \| < 2b$$

holds, and, therefore, $y + cz$ is in U. Hence, we have

$$\Sigma(y + cz) \subset H(x_1 - x_0)$$

and

$$\|y\| \geqslant \max\{\operatorname{Re} L(y) : L \in \Sigma(y + cz)\}$$
$$= \max\{\operatorname{Re} L(y + cz) : L \in \Sigma(y + cz)\} = \|y + cz\|.$$

Generally, when so constructed, U is not a neighborhood of x_0. Yet the numbers b and c obtained above are also applicable for x_0 as we shall now see.

For fixed y and z the function $\psi(\gamma) = \|y + \gamma z\|$ is convex, and, hence, the difference $\phi(\gamma) := \psi(\gamma) - \psi(\gamma + c)$ is monotone nonincreasing. Let now y be such that

$$\|y - x_0\| < b, \qquad \gamma_0 := \|x_2 - x_0\|/\|z\| \qquad \text{and} \qquad y_1 := y + \gamma_0 z.$$

Then

$$\|y_1 - x_2\| = \|y - x_0\|$$

and

$$\|y\| - \|y + cz\| = \psi(0) - \psi(c)$$
$$\geqslant \psi(\gamma_0) - \psi(\gamma_0 + c) = \|y + \gamma_0 z\| - \|y + \gamma_0 z + cz\|$$
$$= \|y_1\| - \|y_1 + cz\| \geqslant 0$$

as was shown in the preceding part of the proof. Thus, the theorem is proved.

By means of this characterization we can now exhibit a class of (P)-spaces.

THEOREM 5.2. *Each* (QP)-*space* X *is a* (P)-*space.*

Proof. Let x_0, x_1 be elements of X such that $\Sigma(x) \subset H(x_1 - x_0)$ for all x in the segment (x_0, x_1). Since X is a (QP)-space there exists by virtue of Theorem 4.4. for every $x \in X$ a neighborhood $U(x)$ such that $\Sigma(y) \subset \Sigma(x)$ for all $y \in U(x)$. The set

$$U := \bigcup_{x \in (x_0, x_1)} U(x)$$

is a neighborhood of (x_0, x_1) with the property that $\Sigma(y) \subset H(x_1 - x_0)$ for every $y \in U$.

Now we define for convex sets a property (P) and show that this property is strongly related to (P)-spaces.

DEFINITION 5.3. Let V be a convex subset of X. Then V has property (P) if for every $x \in V$, $z \in X$ with $x + z \in V$ there exist positive numbers c and b such that $y + cz \in V$ holds for every $y \in V$ with $\|y - x\| < b$.

LEMMA 5.4. *X is a* (P)*-space if and only if the closed unit ball* $B(X) := \{x \in X : \|x\| \leqslant 1\}$ *has property* (P).

Proof. Let X be a (P)-space, $x \in B(X)$ and $z \in X$ such that $x + z \in B(X)$. If x is in the interior of $B(X)$, then there exists $\epsilon > 0$ such that $\{y \in X : \|x - y\| < 2\epsilon\} \subset B(X)$, and with $b := \epsilon$ and $c := \epsilon/\|z\|$ the requirements of (5.3) are fulfilled. Let now x be a boundary point of $B(X)$. Then $x + z \in B(X)$ implies $\|x + z\| \leqslant \|x\|$ and, since X is a (P)-space, there exist positive numbers b and c such that $\|y + cz\| \leqslant \|y\|$ for all y with $\|y - x\| < b$. Therefore, $y + cz$ is in $B(X)$ for all $y \in B(X)$ with $\|y - x\| < b$.

Let now $B(X)$ have property (P) and let x, z be in X such that $\|x + z\| \leqslant \|x\|$. We may assume $x \neq 0$. Define $x_1 := x/\|x\|$ and $z_1 := z/\|x\|$. Then x_1 is in the boundary of $B(X)$ and $x_1 + z_1 \in B(X)$. Hence, there exist $b_1 > 0$ and $c_1 > 0$ such that $y + c_1 z_1 \in B(X)$ for all $y \in B(X)$ with $\|y - x_1\| < b_1$. For y with $\|y\| = 1$, this implies that $\|y + c_1 z_1\| \leqslant 1$. Choose b with $0 < b < 1$ so that $\|y/\|y\| - x_1\| < b_1$ holds for all y with $\|y - x_1\| < b$ and define $c := (1 - b) \cdot c_1$. Then for all y with $\|y - x_1\| < b$ the inequality

$$\|y + cz_1\| = \|y\| \left\| \frac{y}{\|y\|} + \frac{c}{\|y\|} z_1 \right\| \leqslant \|y\|$$

holds, since $c/\|y\| \leqslant c_1$ and $\|y/\|y\| - x_1\| < b_1$. Hence, we obtain finally $\|y + cz\| \leqslant \|y\|$ for all y with $\|y - x\| < b\|x\|$.

Examples of sets with property (P) are linear subspaces, finite dimensional convex polyhedra, and intersections of finite families of half-spaces. Now we are ready to prove the continuity of the metric projection associated with sets having property (P).

THEOREM 5.5. *Let X be a* (P)*-space and V an approximatively compact convex subset with property* (P). *Then the set-valued metric projection P_V associated with V is continuous.*

Proof. Since V is approximatively compact, P_V is usc by a theorem of Singer [14, p. 386]. Suppose that P_V is not lsc. Then there exist $f \in X$, $v_1 \in P_V(f)$ and $\epsilon_1 > 0$ such that the set

$$\{g \in X : P_V(g) \cap B(v_1, \epsilon_1) \neq \varnothing\}$$

is not a neighborhood of f, that is, there exists a sequence f_n which converges to f, such that

$$P_V(f_n) \cap B(v_1, \epsilon_1) = \varnothing \qquad \text{for all } n.$$

Let v_n be elements of $P_V(f_n)$. Since V is approximatively compact, a subsequence v_{n_i} converges to an element $v_2 \in V$, which is in $P_V(f)$.

The closed segment $[v_1, v_2]$ is a subset of $P_V(f)$ since $P_V(f)$ is convex. Define

$$S_1 := \{v \in [v_1, v_2] : \text{For each neighborhood } W \text{ of } v \text{ the set } P_V(f_n) \cap W \text{ is nonvoid for infinitely many } n\}.$$

By construction we have $v_1 \notin S_1$ and $v_2 \in S_1$. Now we show that S_1 is closed. Let v_3 be a cluster point of S_1 and W a neighborhood of v_3. Then there is an element v_4 in S_1 such that W is also neighborhood of v_4. Hence, $P_V(f_n) \cap W \neq \varnothing$ for infinitely many n.

Since S_1 is closed there is a maximal θ in $[0, 1]$ for which

$$v_\theta := v_2 + \theta(v_1 - v_2)$$

is in S_1. By virtue of $v_1 \notin S_1$ and $v_2 \in S_1$ we obtain $\theta < 1$ and

$$z := (1 - \theta)(v_1 - v_2) \neq 0.$$

Now we have $v_1 = v_\theta + z \in V$ and, using $v_1, v_\theta \in P_V(f)$,

$$\| f - v_1 \| = \| f - v_\theta - z \| \leqslant \| f - v_\theta \|.$$

Since V has property (P) there exists $b > 0$ and $c > 0$ such that $v + cz \in V$ for all $v \in V$ with $\| v - v_\theta \| < b$. Since X is a (P)-space, there exist $b > 0$ and $c > 0$ such that $\| g - cz \| \leqslant \| g \|$ for all $g \in X$ with $\| g - (f - v_\theta)\| < b$. (Obviously, we may choose b and c so that they are applicable for both statements.)

Since v_θ is in S_1, there exists a subsequence f_{n_i} and elements w_i in $P_V(f_{n_i})$ such that $\lim w_i = v_\theta$. We may assume $\| w_i - v_\theta \| < b/2$ and $\| f_{n_i} - f \| < b/2$. Then we obtain

$$\|(f_{n_i} - w_i) - (f - v_\theta)\| \leqslant \| f_{n_i} - f \| + \| w_i - v_\theta \| < b,$$

whence it follows that

$$\| f_{n_i} - w_i - cz \| \leqslant \| f_{n_i} - w_i \|.$$

Now we have $w_i + cz \in V$ and $w_i \in P_V(f_{n_i})$, which yields $w_i + cz \in P_V(f_{n_i})$. Therefore, $v_\theta + cz = \lim(w_i + cz)$, and, hence, $v_\theta + cz \in S_1$. This contradicts the assumption that θ was the maximal number such that $v_\theta \in S_1$. Thus, we have proved that P_V is lsc.

Finally, we show by an example that Theorem 5.5 is not correct if the hypothesis that V has property (P) is omitted. We note that this example also shows that the conclusions (a) ⇒ (b) of [8, Theorem 6] and (A) ⇒ (B) of [9, Theorem 2] are not valid.

Let X be the space R^3 provided with the maximum norm

$$\|(x_1, x_2, x_3)\| := \max(|x_1|, |x_2|, |x_3|).$$

For V we take the cone

$$V := \{(x_1, x_2, x_3) \in R^3 : (x_1 + x_3)^2 + x_2^2 \leqslant x_3^2, 0 \leqslant x_3 \leqslant 1\},$$

which fails to have property (P). We determine P_V along the straight line $\{(1, x_2, 0) : x_2 \in R\}$, and obtain that P_V is the whole segment $[(0, 0, 0), (0, 0, 1)]$ for $|x_2| \leqslant 1$, and P_V is just one point of the circle

$$\{(x_1, x_2, 1) \in R^3 : (x_1 + 1)^2 + x_2^2 = 1\}$$

for $|x_2| > 1$. Thus, P_V fails to be continuous.

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